

# A GENERALIZATION OF THE BARBAN-DAVENPORT-HALBERSTAM THEOREM TO NUMBER FIELDS (APPEARED IN *JOURNAL OF NUMBER THEORY*)

ETHAN SMITH

ABSTRACT. For a fixed number field  $K$ , we consider the mean square error in estimating the number of primes with norm congruent to  $a$  modulo  $q$  by the Chebotarëv Density Theorem when averaging over all  $q \leq Q$  and all appropriate  $a$ . Using a large sieve inequality, we obtain an upper bound similar to the Barban-Davenport-Halberstam Theorem.

## 1. INTRODUCTION

The mean square error in Dirichlet's Theorem for primes in arithmetic progressions was first studied by Barban [1] and by Davenport and Halberstam [3, 4]. Bounds such as the following are usually referred to as the Barban-Davenport-Halberstam Theorem, although this particular refinement is attributed to Gallagher. Let

$$\psi(x; q, a) := \sum_{\substack{p^m \leq x, \\ p^m \equiv a \pmod{q}}} \log p.$$

Then, for fixed  $M > 0$ ,

$$\sum_{q \leq Q} \sum_{\substack{a=1, \\ (a,q)=1}}^q \left( \psi(x; q, a) - \frac{x}{\varphi(q)} \right)^2 \ll xQ \log x \quad (1)$$

if  $x(\log x)^{-M} \leq Q \leq x$ . See [5, p. 169]. Here  $\varphi$  is the Euler totient function. The sum on the left may be viewed as the mean square error in the Chebotarëv Density Theorem when averaging over cyclotomic extensions of  $\mathbb{Q}$ .

The inequality in (1) was later refined by Montgomery [12] and Hooley [8], both of whom gave asymptotic formulae valid for various ranges of  $Q$ . See also [2, Theorem 1]. Montgomery's method is based on a result of Lavrik [10] on the distribution of twin primes, while Hooley's method relies on the large sieve. For recent work concerning such asymptotics, see [11].

Results of this type have also been generalized to number fields. Wilson considered error sums over prime ideals falling into a given class of the narrow ideal class group in [13]. While in [7], Hinz considered sums of principal prime ideals given by a generator which is congruent to a given algebraic integer modulo an integral ideal and whose conjugates fall into a designated range.

---

2000 *Mathematics Subject Classification.* 11N36, 11R44.

*Key words and phrases.* large sieve, Chebotarëv Density Theorem, Barban-Davenport-Halberstam Theorem.

Let  $K$  be a fixed algebraic number field which is normal over  $\mathbb{Q}$ . In this paper, we consider the mean square error in the Chebotarëv Density Theorem when averaging over cyclotomic extensions of  $K$ . That is, we consider sums of the form

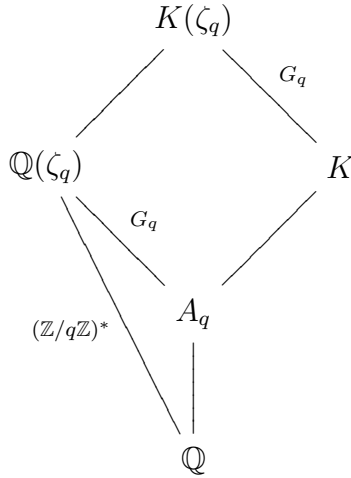
$$\psi_K(x; q, a) := \sum_{\substack{N\mathfrak{p}^m \leq x, \\ N\mathfrak{p}^m \equiv a \pmod{q}}} \log N\mathfrak{p}.$$

Here the sum is over powers of prime ideals of  $K$ , and there is no restriction to principal primes.

For each positive integer  $q$ , we let  $A_q := K \cap \mathbb{Q}(\zeta_q)$ . So,  $A_q$  is an Abelian (possibly trivial) extension of  $\mathbb{Q}$ . We have a natural composition of maps:

$$\text{Gal}(K(\zeta_q)/K) \hookrightarrow \text{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/q\mathbb{Z})^*; \quad (2)$$

and in fact,  $\text{Gal}(K(\zeta_q)/K) \cong \text{Gal}(\mathbb{Q}(\zeta_q)/A_q)$ . We let  $G_q$  denote the image of composition of maps in (2). Then, in particular,  $G_q \cong \text{Gal}(K(\zeta_q)/K)$ . See the diagram below.



Define  $\varphi_K(q) := |G_q|$ . By the Chebotarëv Density Theorem, for each  $a \in G_q$ ,

$$\psi_K(x; q, a) = \sum_{\substack{N\mathfrak{p}^m \leq x, \\ N\mathfrak{p}^m \equiv a \pmod{q}}} \log N\mathfrak{p} \sim \frac{x}{\varphi_K(q)}.$$

**Theorem 1.** For a fixed  $M > 0$ ,

$$\sum_{q \leq Q} \sum_{a \in G_q} \left( \psi_K(x; q, a) - \frac{x}{\varphi_K(q)} \right)^2 \ll xQ \log x$$

if  $x(\log x)^{-M} \leq Q \leq x$ .

*Remark 1.* Note that the above agrees with (1) when  $K = \mathbb{Q}$ . The method of proof is essentially an adaptation of the proof of (1) given in [5, pp. 169-171], the main idea being an application of the large sieve.

*Remark 2.* We say nothing about the constant implied by the symbol  $\ll$  in the present paper. However, the author has recently adapted the methods of Hooley [8] to refine the result into an asymptotic formula. This work is the subject of a forthcoming paper.

*Remark 3.* The above result is unconditional and gives a better bound than the Grand Riemann Hypothesis (GRH). See Section 4 for comparison with GRH.

## 2. PRELIMINARIES AND INTERMEDIATE ESTIMATES

We will use lower case Roman letters for rational integers and Fraktur letters for ideals of the number field  $K$ . In particular,  $p$  will always denote a rational prime and  $\mathfrak{p}$  will always denote a prime ideal in  $\mathcal{O}_K$ , the ring of integers of  $K$ . We also let  $g(K/\mathbb{Q}; p)$  and  $f(K/\mathbb{Q}; p)$  denote the number of primes of  $K$  lying above  $p$  and the degree of any prime of  $K$  lying above  $p$ , respectively. Note that  $f(K/\mathbb{Q}; p)$  is well-defined since  $K$  is normal over  $\mathbb{Q}$ .

Let  $\mathcal{X}(q)$  denote the character group modulo  $q$ ,  $\mathcal{X}^*(q)$  the characters which are primitive modulo  $q$ , and let  $G_q^\perp$  denote the subgroup of characters that are trivial on  $G_q$ . Then the character group  $\widehat{G_q}$  is isomorphic to  $\mathcal{X}(q)/G_q^\perp$ , and the number of such characters is  $\varphi_K(q) = |G_q| = \varphi(q)/|G_q^\perp|$ . As usual, we denote the trivial character of the group  $\mathcal{X}(q)$  by  $\chi_0$ .

For any Hecke character  $\xi$  on the ideals of  $\mathcal{O}_K$ , we define

$$\psi_K(x, \xi) := \sum_{N\mathfrak{a} \leq x} \xi(\mathfrak{a}) \Lambda_K(\mathfrak{a});$$

and for each character  $\chi \in \mathcal{X}(q)$ , we define

$$\psi'_K(x, \chi \circ N) := \begin{cases} \psi_K(x, \chi \circ N), & \chi \not\equiv \chi_0 \pmod{G_q^\perp}, \\ \psi_K(x, \chi \circ N) - x, & \chi \equiv \chi_0 \pmod{G_q^\perp}. \end{cases}$$

Here,  $\Lambda_K$  is the von Mangoldt function defined on the ideals of  $\mathcal{O}_K$ , i.e.,

$$\Lambda_K(\mathfrak{a}) := \begin{cases} \log N\mathfrak{p}, & \mathfrak{a} = \mathfrak{p}^m, \\ 0, & \text{otherwise.} \end{cases}$$

**Lemma 1.**

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi \in \mathcal{X}^*(q)} |\psi_K(x, \chi \circ N)|^2 \ll (x + Q^2)x \log x.$$

*Proof.* For  $n \in \mathbb{N}$ , we define

$$\begin{aligned} D_K(n) &:= \#\{\mathfrak{p}^m \triangleleft \mathcal{O}_K : N\mathfrak{p}^m = n\}; \\ \Lambda_K^*(n) &:= \begin{cases} \log p^{f(K/\mathbb{Q}; p)}, & n = p^k, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Now, note that

$$\psi_K(x, \chi \circ N) = \sum_{N\mathfrak{a} \leq x} \chi(N\mathfrak{a}) \Lambda_K(\mathfrak{a}) = \sum_{n \leq x} \chi(n) D_K(n) \Lambda_K^*(n).$$

We apply the large sieve in the form of Theorem 4 in chapter 27 of [5] to see that

$$\begin{aligned}
\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi \in \mathcal{X}^*(q)} |\psi_K(x, \chi \circ N)|^2 &\ll (x + Q^2) \sum_{n \leq x} (D_K(n) \Lambda_K^*(n))^2 \\
&= (x + Q^2) \sum_{p^k \leq x} g(K/\mathbb{Q}; p) D_K(p^k) \Lambda_K^*(p^k)^2 \\
&\ll (x + Q^2) \sum_{N\mathfrak{p}^m \leq x} \Lambda_K(\mathfrak{p}^m)^2 \\
&\ll (x + Q^2) x \log x.
\end{aligned}$$

□

**Lemma 2.** *If  $\xi_1$  and  $\xi_2$  are Hecke characters modulo  $\mathfrak{q}_1$  and  $\mathfrak{q}_2$  respectively, and if  $\xi_1$  induces  $\xi_2$ , then*

$$\psi_K(x, \xi_2) = \psi_K(x, \xi_1) + O((\log qx)^2),$$

where  $(q) = \mathfrak{q}_2 \cap \mathbb{Z}$ .

*Proof.*

$$\begin{aligned}
|\psi_K(x, \xi_1) - \psi_K(x, \xi_2)| &= \left| \sum_{\substack{N\mathfrak{p}^m \leq x, \\ (\mathfrak{p}, \mathfrak{q}_2) > 1}} \xi_1(\mathfrak{p}^m) \log N\mathfrak{p} \right| \leq \sum_{\substack{p^k \leq x, \\ (p, q) > 1}} D_K(p^k) f(K/\mathbb{Q}; p) \log p \\
&= \sum_{p|q} \sum_{\substack{k=1, \\ f(K/\mathbb{Q}; p)|k}}^{\lfloor \log x / \log p \rfloor} g(K/\mathbb{Q}; p) f(K/\mathbb{Q}; p) \log p \\
&\ll \sum_{p|q} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \ll (\log qx)^2.
\end{aligned}$$

□

**Lemma 3.** *If  $\chi$  is a character modulo  $q \leq (\log x)^{M+1}$ , then there exists a positive constant  $C$  (depending on  $M$ ) such that*

$$\psi'_K(x, \chi \circ N) \ll x \exp \left\{ -C \sqrt{\log x} \right\}.$$

*Proof.* As a Hecke character on the ideals of  $\mathcal{O}_K$ ,  $\chi \circ N$  may not be primitive modulo  $q\mathcal{O}_K$ . Let  $\xi = \xi_\chi$  be the primitive Hecke character which induces  $\chi \circ N$ , and let  $\mathfrak{f}_\chi$  be its conductor. Write  $s = \sigma + it$ . By [9, Theorem 5.35], there exists an effective constant  $c_0 > 0$  such that the Hecke  $L$ -function  $L(s, \xi) := \sum_{N\mathfrak{a} \leq x} \xi(\mathfrak{a})(N\mathfrak{a})^{-s}$  has at most one zero in the region

$$\sigma > 1 - \frac{c_0}{[K : \mathbb{Q}] \log(|d_K| N\mathfrak{f}(|t| + 3))}, \quad (3)$$

where  $d_K$  denotes the discriminant of the number field. Further, if such a zero exists, it is real and simple. In the case that such a zero exists, we call it an “exceptional zero” and

denote it by  $\beta_\xi$ . Thus, by [9, Theorem 5.13], there exists  $c_1 > 0$  such that

$$\psi_K(x, \xi) = \delta_\xi x - \frac{x^{\beta_\xi}}{\beta_\xi} + O\left(x \exp\left\{\frac{-c_1 \log x}{\sqrt{\log x} + \log N\mathfrak{f}_\chi}\right\} (\log(xN\mathfrak{f}_\chi))^4\right),$$

where

$$\delta_\xi = \begin{cases} 1, & \xi \text{ trivial,} \\ 0, & \text{otherwise,} \end{cases}$$

and the term  $x^{\beta_\xi}/\beta_\xi$  is omitted if the  $L$ -function  $L(s, \xi)$  has no exceptional zero in the region (3). Now, since  $\mathfrak{f}_\chi | q\mathcal{O}_K$  and  $q \leq (\log x)^{M+1}$ , we have the following bound on the error term:

$$x \exp\left\{\frac{-c_1 \log x}{\sqrt{\log x} + \log N\mathfrak{f}_\chi}\right\} (\log(xN\mathfrak{f}_\chi))^4 \ll x \exp\left\{-c_2 \sqrt{\log x}\right\}$$

for some positive constant  $c_2$ .

By [6, Theorem 3.3.2], we see that for every  $\epsilon > 0$ , there exists a constant  $c_\epsilon > 0$  such that if  $\beta_\xi$  is an exceptional zero for  $L(s, \xi)$ , then

$$\beta_\xi < 1 - \frac{c_\epsilon}{(N\mathfrak{f}_\chi)^\epsilon} \leq 1 - \frac{c_\epsilon}{q^{[K:\mathbb{Q}]\epsilon}}.$$

Thus,

$$x^{\beta_\xi} < x \exp\left\{-c_\epsilon (\log x) q^{-[K:\mathbb{Q}]\epsilon}\right\} < x \exp\left\{-c_\epsilon (\log x)^{1/2}\right\}$$

upon choosing  $\epsilon$  so that  $[K:\mathbb{Q}]\epsilon = (2M+2)^{-1}$ . Whence, for  $q \leq (\log x)^{M+1}$ , there exists  $C > 0$  such that

$$\psi_K(x, \xi) = \delta_\xi x + O\left(x \exp\left\{-C \sqrt{\log x}\right\}\right).$$

Therefore, by Lemma 2,

$$\psi'_K(x, \chi \circ N) \ll x \exp\left\{-C \sqrt{\log x}\right\}$$

for  $q \leq (\log x)^{M+1}$ . □

### 3. PROOF OF MAIN THEOREM

For  $a \in G_q$ , we define the error term  $E_K(x; q, a) := \psi_K(x; q, a) - \frac{x}{\varphi_K(q)}$ , and note that

$$E_K(x; q, a) = \frac{1}{\varphi_K(q)} \sum_{\chi \in \widehat{G_q}} \bar{\chi}(a) \psi'_K(x, \chi \circ N).$$

Now we form the square of the Euclidean norm and sum over all  $a \in G_q$  to see

$$\begin{aligned}
\sum_{a \in G_q} |E_K(x; q, a)|^2 &= \frac{1}{\varphi_K(q)^2} \sum_{a \in G_q} \left| \sum_{\chi \in \widehat{G_q}} \bar{\chi}(a) \psi'_K(x, \chi \circ N) \right|^2 \\
&= \frac{1}{\varphi_K(q)^2} \sum_{a \in G_q} \sum_{\chi_1 \in \widehat{G_q}} \sum_{\chi_2 \in \widehat{G_q}} \bar{\chi}_1(a) \chi_2(a) \psi'_K(x, \chi_1 \circ N) \overline{\psi'_K(x, \chi_2 \circ N)} \\
&= \frac{1}{\varphi_K(q)} \sum_{\chi \in \widehat{G_q}} |\psi'_K(x, \chi \circ N)|^2 \\
&= \frac{1}{\varphi(q)} \sum_{\chi \in \mathcal{X}(q)} |\psi'_K(x, \chi \circ N)|^2.
\end{aligned}$$

For each  $\chi \in \mathcal{X}(q)$ , we let  $\chi_*$  denote the primitive character which induces  $\chi$ . By Lemma 2, we have  $\psi'_K(x, \chi \circ N) = \psi'_K(x, \chi_* \circ N) + O((\log qx)^2)$ . Hence, summing over  $q \leq Q$  and exchanging each character for its primitive version, we have

$$\sum_{q \leq Q} \sum_{a \in G_q} E_K(x; q, a)^2 \ll \sum_{q \leq Q} (\log qx)^4 + \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \in \mathcal{X}(q)} |\psi'_K(x, \chi_* \circ N)|^2.$$

The first term on the right is clearly smaller than  $xQ \log x$ , so we concentrate on the second. Now,

$$\begin{aligned}
\sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \in \mathcal{X}(q)} |\psi'_K(x, \chi_* \circ N)|^2 &= \sum_{q \leq Q} \sum_{\chi \in \mathcal{X}^*(q)} |\psi'_K(x, \chi \circ N)|^2 \sum_{k \leq Q/q} \frac{1}{\varphi(kq)} \\
&\ll \sum_{q \leq Q} \frac{1}{\varphi(q)} \left( \log \frac{2Q}{q} \right) \sum_{\chi \in \mathcal{X}^*(q)} |\psi'_K(x, \chi \circ N)|^2 \quad (4)
\end{aligned}$$

since  $\sum_{k \leq Q/q} 1/\varphi(kq) \ll \varphi(q)^{-1} \log(2Q/q)$ . See [5, p. 170]. The proof will be complete once we show that (4) is smaller than  $xQ \log x$  for  $Q$  in the specified range.

As with the proof of (1) in [5, pp. 169-171], we consider large and small  $q$  separately. We start with the large values. Since  $\psi'_K(x, \chi \circ N) \ll \psi_K(x, \chi \circ N)$ , by Lemma 1, we have

$$\sum_{U < q \leq 2U} \frac{U}{\varphi(q)} \sum_{\chi \in \mathcal{X}^*(q)} |\psi'_K(x, \chi \circ N)|^2 \ll (x + U^2)x \log x,$$

which implies

$$\sum_{U \leq q \leq 2U} \frac{1}{\varphi(q)} \left( \log \frac{2Q}{q} \right) \sum_{\chi \in \mathcal{X}^*(q)} |\psi'_K(x, \chi \circ N)|^2 \ll (xU^{-1} + U)x \log x \left( \log \frac{2Q}{U} \right)$$

for  $1 \leq 2U \leq Q$ . Summing over  $U = Q2^{-k}$ , we have

$$\begin{aligned}
\sum_{Q_1 \leq q \leq Q} \frac{1}{\varphi(q)} \left( \log \frac{2Q}{q} \right) \sum_{\chi \in \mathcal{X}^*(q)} |\psi'_K(x, \chi \circ N)|^2 &\ll x \log x \sum_{k=1}^{\left\lfloor \frac{\log(Q/Q_1)}{\log 2} \right\rfloor} (x2^k Q^{-1} + Q2^{-k}) \\
&\ll x^2 Q_1^{-1} (\log x) \log Q + xQ \log x \\
&\ll xQ \log x
\end{aligned} \tag{5}$$

if  $x(\log x)^{-M} \leq Q \leq x$  and  $Q_1 = (\log x)^{M+1}$ .

We now turn to the small values of  $q$ . Applying Lemma 3, we have

$$\begin{aligned}
\sum_{q \leq Q_1} \frac{1}{\varphi(q)} \left( \log \frac{2Q}{q} \right) \sum_{\chi \in \mathcal{X}^*(q)} |\psi'_K(x, \chi \circ N)|^2 &\ll Q_1 (\log Q) \left( x \exp \left\{ -c\sqrt{\log x} \right\} \right)^2 \\
&\ll x^2 (\log x)^{-M} \ll xQ \log x.
\end{aligned} \tag{6}$$

Combining (5) and (6), the theorem follows.  $\square$

#### 4. COMPARISON WITH GRH

Using the bound on the analytic conductor of the  $L$ -function  $L(s, \chi \circ N)$  given in [9, p. 129], GRH implies

$$\begin{aligned}
\sum_{q \leq Q} \sum_{a \in G_q} E_K(x; q, a)^2 &= \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \in \mathcal{X}(q)} |\psi'_K(x, \chi \circ N)|^2 \\
&\ll (\sqrt{x} (\log x)^2)^2 \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \in \mathcal{X}(q)} 1 \\
&= xQ (\log x)^4.
\end{aligned}$$

See [9, Theorem 5.15] for this implication of GRH.

#### 5. ACKNOWLEDGMENT

The main theorem of this paper is an estimate required by the author's work in his PhD dissertation. The author is grateful for the guidance of his advisor, Kevin James.

#### REFERENCES

- [1] M.B. Barban. On the distribution of primes in arithmetic progressions “on average”. *Dokl. Akad. Nauk UzSSR*, 5:5–7, 1964. (Russian).
- [2] M. J. Croft. Square-free numbers in arithmetic progressions. *Proc. London Math. Soc. (3)*, 30:143–159, 1975.
- [3] H. Davenport and H. Halberstam. Primes in arithmetic progressions. *Michigan Math. J.*, 13:485–489, 1966.
- [4] H. Davenport and H. Halberstam. Corrigendum: “Primes in arithmetic progression”. *Michigan Math. J.*, 15:505, 1968.
- [5] Harold Davenport. *Multiplicative Number Theory*. Springer-Verlag, New York, 1980.
- [6] Larry Joel Goldstein. A generalization of the Siegel-Walfisz theorem. *Trans. Amer. Math. Soc.*, 149:417–429, 1970.
- [7] Jürgen G. Hinz. On the theorem of Barban and Davenport-Halberstam in algebraic number fields. *J. Number Theory*, 13(4):463–484, 1981.

- [8] Christopher Hooley. On the Barban-Davenport-Halberstam theorem. I. *J. Reine Angew. Math.*, 274/275:206–223, 1975. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, III.
- [9] Henryk Iwaniec and Emmanuel Kowalski. *Analytic Number Theory*, volume 53 of *Colloquium Publications*. American Mathematical Society, Providence, 2004.
- [10] A. F. Lavrik. On the twin prime hypothesis of the theory of primes by the method of I. M. Vinogradov. *Soviet Math. Dokl.*, 1:700–702, 1960.
- [11] H.-Q. Liu. Barban-Davenport-Halberstam average sum and exceptional zero of  $L$ -functions. *J. Number Theory*, 121(4):1044–1059, 2008.
- [12] H. L. Montgomery. Primes in arithmetic progressions. *Michigan Math. J.*, 17:33–39, 1970.
- [13] Robin J. Wilson. The large sieve in algebraic number fields. *Mathematika*, 16:189–204, 1969.

DEPARTMENT OF MATHEMATICAL SCIENCES, CLEMSON UNIVERSITY, BOX 340975 CLEMSON, SC 29634-0975

*E-mail address:* `ethans@math.clemson.edu`

*URL:* `www.math.clemson.edu/~ethans`